Appendix

p-adic Numbers

In Chap. 7 we saw that the moduli space of curves over the complex numbers comes in a natural manner with an arithmetic structure. Just to remind you let me repeat the following facts. If $\mathcal{C}$ is a curve, then it is the set of common zeros of finitely many homogeneous polynomials. For example, a curve in the projective plane can be given by the zeros of one homogeneous polynomial. If after a suitable change of coordinates the polynomials can be given in such a way that all coefficients are rational numbers then we say that the curve $\mathcal{C}$ can be *defined over the rational numbers* $\mathbb{Q}$.

If g is a rational polynomial of degree k we can write it as

$$g = \sum_i r_i m_i, \qquad r_i = \frac{s_i}{t_i}$$

where m_i are monomials (just products of the variables) and the s_i and t_i are integers. If we now multiply g by the least common multiple t of the t_i we get

$$\hat{g} = t \cdot g = \sum_i \hat{s}_i \cdot m_i$$

with $\hat{s}_i$ integers. Of course g and $\hat{g}$ have the same set of points as zeros. Hence every curve defined over the rational numbers $\mathbb{Q}$ can be defined over the integers $\mathbb{Z}$. Hence we can use all number theoretic techniques to get information on the geometric (and arithmetic) structure. (This works also in the opposite direction.)

Now, how do we get from the rational numbers to the complex numbers? Of course this is well known. But the technique is also essential in the case of the p-adic numbers, so let me repeat it briefly. For the rational numbers we have the so-called *absolute value*. It allows us to define a topology and the notion of a convergent sequence. As you know there exist sequences of rational numbers which "converge" but have no limits in $\mathbb{Q}$ (of course strictly speaking we do not talk about convergence in this case). We enlarge $\mathbb{Q}$ by adding these limit points to get the real numbers $\mathbb{R}$.

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One way to construct $\mathbb{R}$ in more precise terms is the following. We call a sequence (x_n) a Cauchy sequence if for every rational number $\epsilon > 0$ there exists a natural number n_0 such that

$$|x_n - x_m| < \epsilon \qquad \text{if } n, m > n_0.$$

As remarked above not every Cauchy sequence has to have a limit. But if it has a limit and this limit is 0, then we call (x_n) a *zero sequence*.

We can define addition, subtraction and multiplication on the set of Cauchy sequences element-wise, for example

$$(x_n) + (y_n) := (x_n + y_n).$$

The real numbers are now defined as the set of classes of Cauchy sequences modulo zero sequences. This means that we identify two Cauchy sequences (x_n) and (z_n) if $(x_n) - (z_n)$ is a zero sequence. All operations can be well defined on these classes. In addition if $[(x_n)]$ is not the zero class (equivalently (x_n) is not a zero sequence) we can define the multiplicative inverse $[(x_n)]^{-1}$ in the following way. By adding a suitable zero sequence to (x_n) we can arrange that for the new sequence (still denoted by (x_n)) $x_n \neq 0$ for all n. In this way we stay in the same class. We set

$$[(x_n)]^{-1} := \left[\left(\frac{1}{x_n}\right)\right]$$

which is again a Cauchy sequence, because the x_n are bounded away from 0.

It is easy to see that $\mathbb{R}$ will be a field and that $\mathbb{Q}$ is embedded as a subfield via the constant sequence

$$[(x_n)] \qquad \text{for all } n : x_n = x \in \mathbb{Q}.$$

Now we can also define the absolute value in $\mathbb{R}$ by setting

$$|[(x_n)]| := [(|x_n|)].$$

With this we see that in $\mathbb{R}$ all Cauchy sequences have a limit. Hence we call $\mathbb{R}$ the completion of $\mathbb{Q}$ with respect to the absolute value $|\,.\,|$.

From the mathematical viewpoint $\mathbb{R}$ is not yet fully satisfying: we cannot solve every algebraic equation. In technical terms $\mathbb{R}$ is not *algebraically closed.* For this reason we have to "add" a single element, the root i of the polynomial equation

$$x^2 + 1 = 0.$$

It obeys the multiplicative law $\mathrm{i}^2 = -1$. In this manner we get the complex numbers $\mathbb{C}$. We can extend the absolute value in $\mathbb{R}$ to the usual complex absolute value. $\mathbb{C}$ is now an algebraically closed complete field.

But what forces us to stick to this special absolute value to start from the rational numbers? Let us list the characteristic features of the absolute value which allowed us to work as above. The absolute value is a map

$$|\,.\,| \ : \mathbb{Q} \to \mathbb{Q}$$

with

$$|a| \geq 0, \quad a \in \mathbb{Q}, \qquad |a| = 0 \quad \text{if and only if} \;\; a = 0. \tag{1}$$

$$|a \cdot b| = |a| \cdot |b|, \quad a, b \in \mathbb{Q} \quad \text{(multiplicative law)} \tag{2}$$

$$|a + b| \leq |a| + |b|, \quad a, b \in \mathbb{Q} \quad \text{(triangle identity)} \tag{3}$$

If we have such a map we call it a *valuation* for the field $\mathbb{Q}$. We can generalize this immediately to arbitrary fields (for the values we remain in $\mathbb{Q}$ or $\mathbb{R}$).

There is one map, the so-called *trivial valuation*, defined by

$$|a| = \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0\ . \end{cases}$$

This valuation is of no interest to us. If we use the word valuation we always assume a nontrivial one.

Let us consider a very interesting one. If

$$r = \frac{n}{m} \in \mathbb{Q}, \quad r \neq 0, \quad n, m \in \mathbb{Z}$$

then we can write

$$n = 2^s n', \quad m = 2^t m'$$

where n' and m' are integers which are not divisible by 2. We set

$$|r|_2 := 2^{t-s} \text{ and } |0|_2 := 0.$$

Let us check the conditions above.
Condition 1: is clear.
Condition 2: Normalized as above we get

$$r_1 = \frac{n_1}{m_1} = 2^l \cdot \frac{n_1'}{m_1'}, \quad r_2 = \frac{n_2}{m_2} = 2^p \cdot \frac{n_2'}{m_2'},$$

$$|r_1|_2 \cdot |r_2|_2 = 2^{-l} 2^{-p} = 2^{-(l+p)}$$

$$r_1 r_2 = 2^{l+p} \cdot \frac{n_1' n_2'}{m_1' m_2'}$$

Because 2 is a prime number it divides neither $n_1'n_2'$ nor $m_1'm_2'$. Hence we get

$$|r_1 r_2|_2 = 2^{-(l+p)}.$$

This is what we had to show.
Condition 3: We keep the same notation as above. We separate two cases.
($l \neq p$): without restriction we assume $l < p$.

$$r_1 + r_2 = 2^l \cdot \left(\frac{n_1'}{m_1'} + 2^{p-l} \frac{n_2'}{m_2'} \right) = 2^l \cdot \frac{n_1' m_2' + 2^{p-l} n_2' m_1'}{m_1' m_2'}.$$

Now 2 divides neither the numerator (because 2 divides only one term in the sum) nor the denominator, hence

$$|r_1 + r_2|_2 = 2^{-l} = |r_1|_2 \leq |r_1|_2 + |r_2|_2.$$

($l = p$): here we get

$$r_1 + r_2 = 2^l \cdot \frac{n_1' m_2' + n_2' m_1'}{m_1' m_2'}.$$

It can happen that 2 divides the numerator, hence

$$|r_1 + r_2|_2 = 2^{-(l+q)} \leq |r_1|_2 \leq |r_1|_2 + |r_2|_2$$

with q a positive integer.
In both cases condition 3 is fulfilled. Even more is true. We can replace (3) by

$$|r_1 + r_2| \leq \max(|r_1|, |r_2|) \tag{3'}$$

and we know in addition

$$|r_1 + r_2| = \max(|r_1|, |r_2|) \quad \text{ if } |r_1| \neq |r_2|$$

If the stronger condition (3′) is valid we call $|\,.\,|$ a *nonarchimedian* valuation. Another name for it is an *ultrametric*. In a nonarchimedian valuation the integers are always within the "unit circle". We can see this as follows. We have $|1| = 1$ as usual ($|a| = |1 \cdot a| = |1| \cdot |a|$). But now

$$|2| = |1 + 1| \leq \max(|1|, |1|) = |1| = 1.$$

By induction we get $|n| \leq 1$ for all integers n. This is a rather strange result in contrast to the usual absolute value.

In the above the only fact we used about the number 2 was that it was a prime number. Hence we can define in an identical manner

$$|a|_p = p^{-s} \quad \text{if } a = p^s \cdot \frac{n'}{m'}, \quad |0|_p = 0$$

where p divides neither n' nor m'. We call $|\,.\,|_p$ the *p-adic valuation*. If there is a danger of confusion we use $|\,.\,|_\infty$ for the usual absolute value.

We can define a topology, convergence, Cauchy sequences and so on. With this we see that all these valuations are essentially different. To see this let us consider the sequence

$$(x_n) = (p^n)$$

for a fixed prime number p. Now

$$|p^n|_\infty = p^n$$

and hence it is unbounded and diverges in the usual topology. On the other hand

$$|p^n|_p = p^{-n}, \qquad |p^n|_q = 1, \quad \text{for } q \neq p.$$

We see (x_n) is a zero sequence in the p-adic topology and a bounded (but not zero) sequence in the q-adic topology.

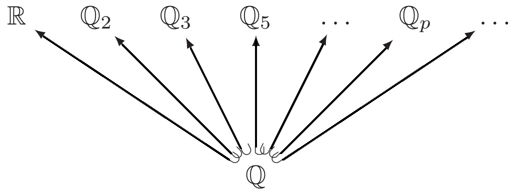
We call two valuations $|\,.\,|_a, |\,.\,|_b$ equivalent if we have

$$|\,.\,|_a = |\,.\,|_b^r$$

with a positive real number r, i.e. if they define the same topology.

Theorem A.1. *(Ostrowski) $|\,.\,|_\infty$ and $|\,.\,|_p$, (p all prime numbers), are representatives of all equivalence classes of valuations for the rational number field $\mathbb{Q}$.*

In the same way as we constructed the real numbers $\mathbb{R}$ via Cauchy sequences with respect to the absolute value we construct the *p-adic numbers* $\mathbb{Q}_p$ starting with the p-adic valuation for every prime number p. Our field $\mathbb{Q}$ lies in all these completions:



In $\mathbb{Q}_p$ we can do analysis similar to real analysis. Just as we can represent every real number as a decimal expansion, we can represent every p-adic number by a p-adic expansion. Take $a \in \mathbb{Q}_p$, by extension of the p-adic valuation to $\mathbb{Q}_p$ we calculate

$$|a|_p = p^{-m}, \quad m \in \mathbb{Z}.$$

With this we can write

$$a = \sum_{k \geq m}^{\infty} a_k p^k = \lim_{n \to \infty} \sum_{k \geq m}^{n} a_k p^k$$

with

$$a_k \in \{0, 1, 2, \ldots, p-1\}.$$

As you can see the sum goes in the opposite direction of the decimal expansion.

In some sense the analysis in $\mathbb{Q}^p$ is much easier than that in $\mathbb{R}$ but more unfamiliar. For example we have in $\mathbb{Q}^p$ the following:

Lemma A.2. *A series $\sum_k a_k$ converges if and only if the sequence (a_k) is a zero sequence.*

(Quite a number of students starting to learn mathematics would be happy if this would work also in $\mathbb{R}$.)

Proof. Let (a_k) be a zero sequence. Then $(m \geq n)$

$$\left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n}^{m} a_k\right| \leq \max_{k=n}^{m}(|a_k|).$$

Because (a_k) is a zero sequence the right-hand side can be made as small as one wants by choosing n big enough. This means that the sequence of partial sums is a Cauchy sequence and hence has a limit. The other direction is the usual argument of real analysis. □

Now we can also define p-adic power series like the *exponential*

$$E(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and the *logarithm*

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}.$$

Despite the fact that the expressions are the same you might expect that the domain of convergence will be different. We write

$$|x|_p = p^{-\operatorname{ord}_p(x)}$$

to define the *p-adic order*. With this we can calculate that $E(x)$ is convergent if and only if $\operatorname{ord}_p(x) > \frac{1}{p-1}$ and $\log(1+x)$ is convergent if and only if $\operatorname{ord}_p(x) > 0$.

This is again a rather strange behaviour. If $p \neq 2$ and x is an integer then $E(x)$ is a convergent series if and only if x is a multiple of p. If $p = 2$ and everything else stays the same, then x has to be a multiple of 4.

One can now introduce p-adic measures, distributions, integrals and Fourier transformations.[1] Let me mention that in $\mathbb{Q}_p$ there is in contrast to $\mathbb{R}$ still a lot of number theory involved. For example, we have also the *p-adic integers* which are defined to be the $x \in \mathbb{Q}_p$ with $\operatorname{ord}_p(x) \geq 0$. Like in the case of the rational numbers we can write every element of $\mathbb{Q}_p$ as quotient of two p-adic integers.

We are still not at the end in our construction of the analogue of $\mathbb{C}$. To get from $\mathbb{R}$ to $\mathbb{C}$ we took the algebraic closure of $\mathbb{R}$. This can also be done with $\mathbb{Q}_p$. In this case we have to "add" infinitely many roots of polynomials to get

[1] See Koblitz, [Ko], p. 30.

to $\mathbb{Q}_p^{a.c.}$. Of course this $\mathbb{Q}_p^{a.c}$ also admits an extension of the p-adic valuation. But in contrast to $\mathbb{C}$ it is no longer complete. We have to again construct the completion of this field. The resulting field is algebraically closed and complete.

Of course, one might ask why one should bother about the p-adic numbers if one is not interested in number theory. Some idea of their importance outside of number theory might be given by the following product formula for all rational numbers $x \neq 0$:

$$|x|_\infty \cdot \prod_{p\in\mathbb{P}} |x|_p = 1.$$

(We use $\mathbb{P}$ to denote the set of all prime numbers.) It says, if we know all p-adic values of a rational number we also know its absolute value. This is a reformulation of the trivial fact that we know the absolute value of a rational number if we know how often all primes appear in the numerator and in the denominator. But there is more behind it. The mathematical idea is that the rational numbers are the objects of primary interest. But they are very difficult to handle from the arithmetic viewpoint. $\mathbb{R}$ and all $\mathbb{Q}_p$ are much easier. Each of them reflects one facet of the complexity of $\mathbb{Q}$. From this viewpoint one calls $\mathbb{Q}$ a *global* field and $\mathbb{R}$ and the different $\mathbb{Q}_p$ *local* fields. If $x \in \mathbb{Q}$ is a number fulfilling a suitable relation (for example to be a zero of a polynomial equation with integral coefficients) then x considered as element in $\mathbb{R}$ and $\mathbb{Q}_p$ clearly fulfils the transformed relation in $\mathbb{R}$, resp. in $\mathbb{Q}_p$. The converse problem is the one that really matters. If we can solve the transformed relations by elements x_∞ and $x_p,\ p \in \mathbb{P}$ (which in most cases is easier) we can ask under which conditions we can assemble this information to get a solution for the starting relation in $\mathbb{Q}$. (This is also known under the name *local–global principle*.)

In the above you might get the (right) feeling that it is better to consider the real number field and all p-adic number fields simultaneously. This is formalized in the concept of *adèles*. We start with the infinite product

$$\mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Q}_p.$$

It consists of all infinite sequences

$$(x_\infty, x_2, x_3, x_5, \ldots, x_p, \ldots)$$

where $x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p$. Now we consider the subset with

$$|x|_p \leq 1$$

for almost every p (which is shorthand for: the condition can be violated only by a finite number of primes). This subset is called the set of *adèles* and is denoted by $\mathbb{A}_\mathbb{Q}$. It is a ring if we define addition and multiplication componentwise. The set of multiplicative invertible elements in $\mathbb{A}_\mathbb{Q}$ is called *idèles.* $\mathbb{Q}$ is now embedded diagonally into $\mathbb{A}_\mathbb{Q}$ by

$$x \mapsto (x, x, x, \ldots).$$

At first sight this is only a formal tool. But because this is a ring we can do algebraic geometry with it and other nice things. The following is an example taken from Manin (details can be found there [Man]). One can define the group $\mathrm{Sl}\,(2, \mathbb{A}_{\mathbb{Q}})$ in a suitable manner. By diagonal embedding, $\mathrm{Sl}\,(2, \mathbb{Q})$ is a discrete subgroup. Now the factor group is a compact group on which we can integrate. By normalizing the measure and splitting it up into the factors corresponding to factor groups of $\mathrm{Sl}\,(2, \mathbb{R})$ and $\mathrm{Sl}\,(2, \mathbb{Q}_p)$ we get the result

$$1 = \frac{\pi^2}{6} \cdot \prod_{p \in \mathbb{P}} (1 - p^{-2}).$$

This relates the arithmetically defined product expression with the transcendental number π. The product is an object which is defined by knowing the measure on the p-adic groups and it determines the analogous measure on the real group. Of course the above formula is not a new result gained only by the use of adèles. But the adèles gives a new insight. Classically it comes from the *Riemann zeta function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

where we have

$$\zeta(2) = \frac{\pi^2}{6} = \prod_{p \in \mathbb{P}} (1 - p^{-2})^{-1}$$

which is the above relation.

Up to now we have done everything over the rational numbers. In fact this is not sufficient. We have to allow finite algebraic field extensions of $\mathbb{Q}$. We can obtain these in the following way. We embed $\mathbb{Q}$ in $\mathbb{C}$ and take an $\alpha \in \mathbb{C}$ which is a root of a rational irreducible polynomial f. (Here irreducible means it does not have a factorization into two nonconstant rational polynomials.) The imaginary unit i for example is such an α. It is the zero of the polynomial

$$\mathbf{X}^2 + 1.$$

Another example is

$$\alpha = \zeta_3 = \exp\left(\frac{2\pi \mathrm{i}}{3}\right),$$

a so-called 3rd root of unity. It is a zero of the polynomial

$$\mathbf{X}^2 + \mathbf{X} + 1.$$

We denote the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and α by $\mathbb{Q}(\alpha)$. It is also a finite-dimensional $\mathbb{Q}$-vector space. Its basis is given by

$$1, \alpha, \alpha^2, \ldots, \alpha^m, \quad m = \deg f - 1.$$

Such a field is called a number field. Just as we can build $\mathbb{Q}$ out of the integers as their quotients, we can find "integers" in $\mathbb{Q}(\alpha)$ which do the job here. In contrast to the usual integers , the *rational integers*, these are called *algebraic integers*. With these algebraic integers we can do arithmetic, define divisibility and so on. Unfortunately we have in general no unique factorization of every algebraic integer into primes. Even worse, we have to drop the notion of prime number at all. But there is a rather useful substitute for the prime numbers. The prime numbers of $\mathbb{Z}$ are in a 1:1 correspondence with the classes of nonarchimedian valuations of $\mathbb{Q}$. Now we ask for such nonarchimedian valuations for $\mathbb{Q}(\alpha)$. Let $|\,.\,|$ be such a valuation. By restricting it to $\mathbb{Q}$ we get one of the p-adic valuations. Conversely one can show that for every p-adic valuation on $\mathbb{Q}$, there exist only finitely many extensions to the whole field $\mathbb{Q}(\alpha)$. By embedding into $\mathbb{C}$ the absolute value is already extended to the usual complex absolute value on $\mathbb{Q}(\alpha)$. Different extensions of the absolute value correspond to different embeddings. There are also finitely many of them.

Now we can do everything (completion, adèles, ...) we did for $\mathbb{Q}$ for this number field $\mathbb{Q}(\alpha)$. The corresponding complete field is a more tractable subfield of the big p-adic algebraically closed and complete field.

Hints for Further Reading

Further details can be found in

[Ko] Koblitz, N., *p-adic Numbers, p-adic Analysis and Zeta Functions*, Springer, 1977.

It contains the basic concepts including p-adic integration.
Of course you also find the basics in other books like

[Ba] Bachmann, G., *Introduction to p-adic Numbers and Valuation Theory*, Academic Press, New York 1964.

[Wa] Van der Waerden, B.L., *Algebra II*, Springer, 1967.

Adèles can be found in

[Ta] Tamagawa, T., Adèles, *Proc. Symp. Pure Math.* **9** (1966) 113–121, Amer. Math. Soc., Providence.

[Wei] Weil, A., *Basic Number Theory (3rd rev.ed.)*, Grundl. Math. Wiss. 144, Springer, 1974.

Concerning the more speculative aspects of application of p-adic numbers to string theory, see

[Man] Manin, Y.I., Reflections on Arithmetical Physics, Talk at Poiana-Brasov School on Strings and Conformal Field Theory, 1–14. September. 1987, (appeared in) *Perspect. Phys.*, Academic Press, Boston, MA, 1989.

Since the first edition of this book, p-adic and adelic mathematical physics developed further in the context of p-adic string theory and cosmology. For these more recent developments see

[BF] Brekke, L., Freund, P.G.O., p-adic Numbers in Physics, *Phys. Rep.* **233** (1993) 1–66.

[DDNV] Djordjević, G.S., Dragovich, B., Nesić, L., Volovich, I.V., p-adic and Adelic Minisuperspace Quantum Cosmology, *Int. J. Mod. Phys.*, A **17** (2002) 1413–1433.

[KRV] Khrennikov, A.Y., Racić, Z., Volovich I.V. (eds), *p-adic Mathematical Physics, 2nd International Conference*, Belgrade, Serbia, and Montenegro, 15–21.9.2005, AIP, Melville, New York, 2006.

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